HW. # 4

Homework problems are taken from “Principles of Mathematical Analysis” by W. Rudin and “Real Analysis” by N. L. Carothers. The problems are color coded to indicate level of difficulty. The color green indicates an elementary problem, which you should be able to solve effortlessly. Yellow means that the problem is somewhat harder. Red indicates that the problem is hard. You should attempt the hard problems especially.

Unless the contrary is explicitly stated, all numbers that are mentioned in these exercises are understood to be real.

1. The set $C_r(x) = \{y \in M : \ d(x, \ y) \leq r\}$ is called the *closed ball* about x of radius r. Show that $C_r(x)$ is a closed set, but give an example showing that $C_r(x)$ need not equal the closure of the open ball $B_r(x)$.

2. Show that A is open if and only if $A^\circ = A$ and that A is closed if and only if $\overline{A} = A$.

3. Given a nonempty bounded subset E of **R**, show that sup(E) and inf(E) are elements of $\overline{E}$. Thus sup(E) and inf(E) are elements of E whenever E is *closed*.

4. Show that $diam(A) = diam(\overline{A})$.

5. If $A \subset B$, show that $\overline{A} \subset \overline{B}$. Does $\overline{A} \subset \overline{B}$ imply $A \subset B$? Explain.

6. If A and B are any sets in M, show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$ and $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$. Give an example showing that this last inclusion can be proper.

7. True or False? $(A \cup B)^\circ = A^\circ \cup B^\circ$.

8. Show that $\overline{A} = \left[\text{int}\left(A^c\right)\right]^c$ and that $A^\circ = \left[cl\left(A^c\right)\right]^c$.

9. A set that is simultaneously open and closed is sometimes called a **clopen** set. Show that **R** has no nontrivial clopen sets.

10. Let (M, d) be a metric space and $A \subset M$. Show that if x is a limit point of A, then every neighborhood of x contains infinitely many points of A.

11. Suppose that $x_n \xrightarrow{d} x \in M$, and let $A = \{x\} \cup \{x_n : n \geq 1\}$. Prove that A is closed.

12. Show that any ternary decimal of the form $0.a_1a_2...a_n11$ (base 3), i.e., any finite-length decimal ending in two (or more) 1s, is *not* an element of Δ.

13. Show that Δ contains no (nonempty) open intervals. In particular, show that if x, y ∈ Δ with x < y, then there is some z ∈ [0, 1]\ Δ with x < z < y. (It follows from this that Δ is *nowhere dense*, which is another way of saying that Δ is "small")

14. The endpoints of Δ are those points in Δ having a finite-length base 3 decimal expansion (not necessarily in the proper form), that is, all of the points in Δ of the form $a/3^n$ for some integers n and $0 \le a \le 3^n$. Show that the endpoints of Δ other than 0 and 1 can be written as $0.a_1a_2...a_{n+1}$ (base 3), where each a_k is 0 or 2, except a_{n+1}, which is either 1 or 2. That is, the discarded "middle third" intervals are of the form $(0.a_1a_2...a_n1,\ 0.a_1a_2...a_n2)$, where both entries are points of Δ written in base 3.

15. Show that Δ is *perfect*; that is, every point in Δ is the limit point of a sequence of distinct points from Δ. In fact, show that every point in Δ is the limit of a sequence of distinct endpoints.

16. Let $f : \Delta \to [0,1]$ be the Cantor function and let x, y ∈ Δ with x < y. Show that $f(x) \le f(y)$. If $f(x) = f(y)$, show that x has two distinct binary decimal expansions. Finally, show that $f(x) = f(y)$ if and only if x and y are "consecutive" endpoints of the form $x = 0.a_1a_2...a_n1$ and $y = 0.a_1a_2...a_n2$ (base 3).